

ON SOME MULTICOLOUR RAMSEY PROPERTIES OF RANDOM GRAPHS

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ABSTRACT. The size-Ramsey number $\hat{R}(F)$ of a graph F is the smallest integer m such that there exists a graph G on m edges with the property that any colouring of the edges of G with two colours yields a monochromatic copy of F . In this paper, first we focus on the size-Ramsey number of a path P_n on n vertices. In particular, we show that $5n/2 - 15/2 \leq \hat{R}(P_n) \leq 74n$ for n sufficiently large. (The upper bound uses expansion properties of random d -regular graphs.) This improves the previous lower bound, $\hat{R}(P_n) \geq (1 + \sqrt{2})n - O(1)$, due to Bollobás, and the upper bound, $\hat{R}(P_n) \leq 91n$, due to Letzter. Next we study long monochromatic paths in edge-coloured random graph $\mathcal{G}(n, p)$ with $pn \rightarrow \infty$. Let $\alpha > 0$ be an arbitrarily small constant. Recently, Letzter showed that a.a.s. any 2-edge colouring of $\mathcal{G}(n, p)$ yields a monochromatic path of length $(2/3 - \alpha)n$, which is optimal. Extending this result, we show that a.a.s. any 3-edge colouring of $\mathcal{G}(n, p)$ yields a monochromatic path of length $(1/2 - \alpha)n$, which is also optimal. In general, we prove that for $r \geq 4$ a.a.s. any r -edge colouring of $\mathcal{G}(n, p)$ yields a monochromatic path of length $(1/r - \alpha)n$. We also consider a related problem and show that for any $r \geq 2$, a.a.s. any r -edge colouring of $\mathcal{G}(n, p)$ yields a monochromatic connected subgraph on $(1/(r-1) - \alpha)n$ vertices, which is also tight.

1. INTRODUCTION

Following standard notations, we write $G \rightarrow (F)_r$ if any r -edge colouring of G (that is, any colouring of the edges of G with r colours) yields a monochromatic copy of F . For simplicity, we often write $G \rightarrow F$ instead of $G \rightarrow (F)_2$. Furthermore, we define the *size-Ramsey number* of F as $\hat{R}(F, r) = \min\{|E(G)| : G \rightarrow (F)_r\}$ and again, for simplicity, $\hat{R}(F) = \hat{R}(F, 2)$.

We consider the size-Ramsey number of the path P_n on n vertices. It is obvious that $\hat{R}(P_n) = \Omega(n)$ and that $\hat{R}(P_n) = O(n^2)$ (for example, $K_{2n} \rightarrow P_n$), but the exact behaviour of $\hat{R}(P_n)$ was not known for a long time. In fact, Erdős [13] offered \$100 for a proof or disproof that

$$\hat{R}(P_n)/n \rightarrow \infty \quad \text{and} \quad \hat{R}(P_n)/n^2 \rightarrow 0.$$

This problem was solved by Beck [2] in 1983 who, quite surprisingly, showed that $\hat{R}(P_n) < 900n$. (Each time we refer to inequality such as this one, we mean that the inequality holds for sufficiently large n .) A variant of his proof, provided by Bollobás [9], gives $\hat{R}(P_n) < 720n$. Very recently, the authors of this paper [12] used a different and more elementary argument that shows that $\hat{R}(P_n) < 137n$. The argument was subsequently tuned by

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Letzter [27] who showed that $\hat{R}(P_n) < 91n$. On the other hand, the first nontrivial lower bound was provided by Beck [3] and his result was subsequently improved by Bollobás [7] who showed that $\hat{R}(P_n) \geq (1 + \sqrt{2})n - O(1)$.

In Section 2, we show that for any $r \in \mathbb{N}$, $\hat{R}(P_n, r) \geq \frac{(r+3)r}{4}n - O(r^2)$ (Theorem 2.1), which slightly improves the lower bound of Bollobás [7] for two colours and generalizes it to more colours. It follows that $\hat{R}(P_n) \geq 5n/2 - O(1)$. In Section 3, using expansion properties of random d -regular graphs, we show that $\hat{R}(P_n) \leq 74n$ (Theorem 3.6) which improves the leading constant provided by Letzter [27]. We also generalize our upper bound to more colours, showing that $\hat{R}(P_n, r) \leq 33r4^r n$ (Theorem 3.8).

In Section 4, we deal with the following, closely related problem. It is known, due to Gerencsér and Gyárfás [19], that $K_n \rightarrow P_{(2/3+o(1))n}$; due to Gyárfás, Ruszinkó, Sárközy, and Szemerédi [21, 22] and also Figaj and Łuczak [16], we know that $K_n \rightarrow (P_{(1/2+o(1))n})_3$. Moreover, these results are best possible. Unfortunately, very little is known about the behaviour for more colours; although it is conjectured that $K_n \rightarrow (P_{(1/(r-1)+o(1))n})_r$ for $r \in \mathbb{N} \setminus \{1, 2\}$, which would be best possible. Clearly, if for some subgraph G of K_n , $G \rightarrow P_{cn}$, then $K_n \rightarrow P_{cn}$ as well. On the other hand, one could expect that sparse subgraphs of K_n “arrow” much shorter paths. However, this intuition seems to be false. As a matter of fact, for two colours Letzter [27] showed that a.a.s. $\mathcal{G}(n, p) \rightarrow P_{(2/3-\alpha)n}$, provided that $pn \rightarrow \infty$, which is optimal. (Here and later on, $\alpha > 0$ is an arbitrarily small constant.) We adjust her approach (using also some ideas of Figaj and Łuczak [16]) and show that a.a.s. $\mathcal{G}(n, p) \rightarrow (P_{(1/2-\alpha)n})_3$, provided that $pn \rightarrow \infty$, which is also optimal. For any $r \in \mathbb{N} \setminus \{1, 2, 3\}$ we prove that a.a.s. $\mathcal{G}(n, p) \rightarrow (P_{(1/r-\alpha)n})_r$, provided that $pn \rightarrow \infty$ (Theorem 4.1). This is, perhaps, not sharp but it is a consequence of the poor current understanding of the behaviour for K_n . On the other hand, note that the best one can hope for is that a.a.s. $\mathcal{G}(n, p) \rightarrow (P_{(1/(r-1)+o(1))n})_r$, provided that $pn \rightarrow \infty$, since there are r -colourings of the edges of K_n (and so also of $\mathcal{G}(n, p)$) with no monochromatic path of length $n/(r-1)$.

In the next section, Section 5, we continue with similar direction but relax the property of having P_{cn} as a subgraph to having a component of size cn . It is known, due to Gyárfás [20] and Füredi [18], that for any r -colouring of the edges of K_n , there is a monochromatic component of order $(1/(r-1) + o(1))n$. Moreover, this is best possible if $r-1$ is a prime power. We show that K_n and $\mathcal{G}(n, p)$ behave very similarly with respect to the size of the largest monochromatic component. More precisely, we prove that a.a.s. for any r -colouring of the edges of $\mathcal{G}(n, p)$, there is a monochromatic component of order $(1/(r-1) - \alpha)n$, provided that $pn \rightarrow \infty$ (Theorem 5.3). As before, this result is clearly best possible.

2. LOWER BOUND ON THE SIZE-RAMSEY NUMBER OF P_n

In this section, we improve the lower bound (for two colours) given by Bollobás [7] who showed that $\hat{R}(P_n) \geq (1 + \sqrt{2})(n-1) - 4$. In our result, the leading constant $(1 + \sqrt{2})$ is increased to $5/2$. Moreover, we provide a more general result that holds for any number of colours r , which improves the trivial lower bound $\hat{R}(P_n, r) \geq (r-1)(n-1) + 1$.

Theorem 2.1. *Let $r \geq 1$. Then, for all sufficiently large n*

$$\hat{R}(P_n, r) \geq \frac{(r+3)r}{4}n - \frac{r(5r+11)}{4} + 3.$$

We will need the following auxiliary claim.

Claim 2.2. *Let $k \in \mathbb{N} \cup \{0\}$ and T be a tree. Then, at least one of the following two properties holds:*

- (i) T has k edges e_1, e_2, \dots, e_k such that $T - \{e_1, e_2, \dots, e_k\}$ contains no P_n ,
- (ii) T contains $(k + 2)$ vertex-disjoint connected subgraphs of order at least $\lfloor n/2 \rfloor$ each.

Proof. We prove the statement by induction on k . For $k = 0$, if (i) fails, then T contains a copy of P_n and we are done. Indeed, after splitting the path as equally as possible we get two components of the desired order so (ii) holds.

Let $k \in \mathbb{N} \cup \{0\}$ and suppose that the statement holds for any integer i satisfying $0 \leq i \leq k$. Again, assume that (i) fails for $(k + 1)$; that is, for any choice of e_1, e_2, \dots, e_{k+1} , $T - \{e_1, e_2, \dots, e_{k+1}\}$ contains P_n . We will show that (ii) must hold; that is, T contains $(k + 3)$ vertex-disjoint connected subgraphs of order at least $\lfloor n/2 \rfloor$ each.

Clearly $T \supseteq P_n$. Hence, let e be such that $T - e$ consists of two components, T_1 and T_2 , each of order at least $\lfloor n/2 \rfloor$. By assumption we made (that (i) fails for $(k + 1)$), for any choice of k_1 edges e_1, e_2, \dots, e_{k_1} in T_1 and k_2 edges f_1, f_2, \dots, f_{k_2} in T_2 such that $k_1 + k_2 = k$, either $T_1 - \{e_1, e_2, \dots, e_{k_1}\}$ or $T_2 - \{f_1, f_2, \dots, f_{k_2}\}$ contains P_n .

If $T_1 - \{e_1, e_2, \dots, e_{k_1}\} \supseteq P_n$ and $T_2 - \{f_1, f_2, \dots, f_{k_2}\} \supseteq P_n$ for any choice of the edges, then (by inductive hypothesis) T_1 and T_2 have, respectively, $(k_1 + 2)$ and $(k_2 + 2)$ vertex-disjoint connected subgraphs of size $\lfloor n/2 \rfloor$, giving $k_1 + k_2 + 4 \geq k + 3$ vertex-disjoint connected subgraphs of order $\lfloor n/2 \rfloor$ in T . Therefore, without loss of generality, we may assume that $T_2 - \{f_1, f_2, \dots, f_{k_2}\} \not\supseteq P_n$ for some choice of f_1, f_2, \dots, f_{k_2} , where k_2 is as small as possible. Of course, this implies that $T_1 - \{e_1, e_2, \dots, e_{k_1}\} \supseteq P_n$ for any choice of the edges. Now, we need to consider two cases. If $k_2 = 0$, then (by inductive hypothesis) T_1 has $(k_1 + 2)$ vertex-disjoint connected subgraphs of order $\lfloor n/2 \rfloor$ which, together with T_2 yield $(k + 3)$ desired large subgraphs in T . On the other hand, if $k_2 \geq 1$, then (due to minimality of k_2) we infer that for any choice of $f_1, f_2, \dots, f_{k_2-1}$, $T_2 - \{f_1, f_2, \dots, f_{k_2-1}\} \supseteq P_n$. Thus, (again, by inductive hypothesis) T has $(k_1 + 2) + (k_2 - 1 + 2) = k + 3$ vertex-disjoint connected subgraphs of order $\lfloor n/2 \rfloor$, as needed. \square

Now, we are ready to prove the main result of this section. The proof is based on ideas from the proof from [7] and [3].

Proof of Theorem 2.1. We prove the statement by induction on r . For $r = 1$ the desired inequality is trivially true: $\hat{R}(P_n, 1) \geq n - 1$. Assume that the statement holds for some $r \in \mathbb{N}$ and, for a contradiction, suppose that it fails for $(r + 1)$, that is,

$$\hat{R}(P_n, r + 1) < \frac{(r + 4)(r + 1)}{4}n - \frac{(r + 1)(5(r + 1) + 11)}{4} + 3.$$

Let $G = (V, E)$ be a graph of order N and size $\hat{R}(P_n, r + 1)$, such that $G \rightarrow (P_n)_{r+1}$. Clearly, G is connected. We will independently deal with two cases, depending on N .

Case 1: $N > (r + 2)(n - 3)/2$. Let T be any spanning tree of G . We apply Claim 2.2 with $k = r$. First, let us assume that property (i) in the claim holds; that is, T has r edges e_1, e_2, \dots, e_r such that $T - \{e_1, e_2, \dots, e_r\}$ contains no P_n . We colour all $(N - 1) - r$ edges

in $T - \{e_1, e_2, \dots, e_r\}$ using the first colour. The number of uncoloured edges is at most

$$\begin{aligned} & \hat{R}(P_n, r+1) - (N - r - 1) \\ & < \frac{(r+4)(r+1)}{4}n - \frac{(r+1)(5(r+1)+11)}{4} + 3 - \frac{r+2}{2}(n-3) + r+1 \\ & = \frac{(r+3)r}{4}n - \frac{r(5r+11)}{4} + 3 \leq \hat{R}(P_n, r), \end{aligned}$$

where the last inequality follows from the inductive hypothesis. Thus, we can colour the uncoloured edges with the remaining r colours in such a way that there is no monochromatic P_n . Consequently, $G \not\rightarrow (P_n)_{r+1}$, which gives us the desired contradiction.

Assume then that property (ii) in the claim holds; that is, T contains $(r+2)$ vertex-disjoint connected subgraphs of order at least $\lfloor n/2 \rfloor$ each. We colour $\lfloor n/2 \rfloor - 1$ edges of each of the $(r+2)$ components with the first colour. (If some component has more than $\lfloor n/2 \rfloor - 1$ edges, we select edges to colour arbitrarily.) The number of uncoloured edges is at most

$$\begin{aligned} & \hat{R}(P_n, r+1) - (r+2) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \\ & < \frac{(r+4)(r+1)}{4}n - \frac{(r+1)(5(r+1)+11)}{4} + 3 - (r+2) \left(\frac{n}{2} - \frac{3}{2} \right) \\ & = \frac{(r+3)r}{4}n - \frac{r(5r+11)}{4} + 3 - (r+1) < \hat{R}(P_n, r), \end{aligned}$$

and this yields a contradiction ($G \not\rightarrow (P_n)_{r+1}$), as before.

Case 2: $N \leq (r+2)(n-3)/2$. Let $U \subseteq V$ be any set of size $|U| = n-1$, and let W_1, W_2, \dots, W_r be an equipartition of $V \setminus U$. Clearly, for any $1 \leq i \leq r$,

$$|W_i| \leq \left\lceil \frac{1}{r} \left(\frac{r+2}{2}(n-3) - (n-1) \right) \right\rceil = \left\lceil \frac{n-3}{2} - \frac{2}{r} \right\rceil < \frac{n-1}{2} - \frac{2}{r}.$$

Let G_i be a bipartite subgraph of G induced by the edges between W_i and $W_{i+1} \cup \dots \cup W_r \cup U$. We colour the edges of G_i with the i -th colour and the remaining edges (inside U or W_i 's) with the last colour. Clearly there is no monochromatic (or, in fact, any) copy of P_n in U or W_i 's. Furthermore, each path in G_i must alternate between W_i and $W_{i+1} \cup \dots \cup W_r \cup U$. Thus, the longest path in G_i has at most $2|W_i| + 1 < n$ vertices. We get the desired contradiction ($G \not\rightarrow (P_n)_{r+1}$) for the last time and the proof is finished. \square

3. UPPER BOUND ON THE SIZE-RAMSEY NUMBER OF P_n

In this section, we present various upper bounds on $\hat{R}(P_n)$. Corresponding theorems use different approaches and different probability spaces. Subsection 3.1 uses the existing lemma of Letzter. In Subsection 3.2, another improvement is developed, which gives the strongest bound. Finally, Subsection 3.3 deals with more colours.

Let us recall a few classic models of random graphs that we study in this section and later on in the paper. The *binomial random graph* $\mathcal{G}(n, p)$ is the random graph G with vertex set $[n] := \{1, 2, \dots, n\}$ in which every pair $\{i, j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p . The *binomial random bipartite graph* $\mathcal{G}(n, n, p)$ is the random bipartite graph $G = (V_1 \cup V_2, E)$ with partite sets V_1, V_2 , each of order n , in which

every pair $\{i, j\} \in V_1 \times V_2$ appears independently as an edge in G with probability p . Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity.

Recall that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. Since we aim for results that hold a.a.s., we will always assume that n is large enough. For simplicity, we do not round numbers that are supposed to be integers either up or down; this is justified since these rounding errors are negligible to the asymptomatic calculations we will make. Finally, we use $\log n$ to denote natural logarithms.

However, our main results in this section refer to another probability space, the probability space of random d -regular graphs with uniform probability distribution. This space is denoted $\mathcal{G}_{n,d}$, and asymptotics are for $n \rightarrow \infty$ with $d \geq 2$ fixed, and n even if d is odd.

Instead of working directly in the uniform probability space of random regular graphs on n vertices $\mathcal{G}_{n,d}$, we use the *pairing model* (also known as the *configuration model*) of random regular graphs, first introduced by Bollobás [6], which is described next. Suppose that dn is even, as in the case of random regular graphs, and consider dn points partitioned into n labelled buckets v_1, v_2, \dots, v_n of d points each. A *pairing* of these points is a perfect matching into $dn/2$ pairs. Given a pairing P , we may construct a multigraph $G(P)$, with loops allowed, as follows: the vertices are the buckets v_1, v_2, \dots, v_n , and a pair $\{x, y\}$ in P corresponds to an edge $v_i v_j$ in $G(P)$ if x and y are contained in the buckets v_i and v_j , respectively. It is an easy fact that the probability of a random pairing corresponding to a given simple graph G is independent of the graph, hence the restriction of the probability space of random pairings to simple graphs is precisely $\mathcal{G}_{n,d}$. Moreover, it is well known that a random pairing generates a simple graph with probability asymptotic to $e^{-(d^2-1)/4}$ depending on d , so that any event holding a.a.s. over the probability space of random pairings also holds a.a.s. over the corresponding space $\mathcal{G}_{n,d}$. For this reason, asymptotic results over random pairings suffice for our purposes. For more information on this model, see, for example, the survey of Wormald [33].

Also, we will be using the following well-known concentration inequality. Let $X \in \text{Bin}(n, p)$ be a random variable with the binomial distribution with parameters n and p . Then, a consequence of Chernoff's bound (see, for example, [23, Corollary 2.3]) is that

$$\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq 2 \exp \left(-\frac{\varepsilon^2 \mathbb{E}X}{3} \right)$$

for $0 < \varepsilon < 3/2$.

3.1. Existing approach. Using the following (deterministic) lemma Letzter showed that $\hat{R}(P_n) < 91n$.

Lemma 3.1 ([27]). *Let G be a graph of order cn for some $c > 2$. Assume that for every two disjoint sets of vertices S and T such that $|S| = |T| = n(c-2)/4$ we have $e(S, T) \neq 0$. Then, $G \rightarrow P_n$.*

In fact, she showed that a.a.s. $\mathcal{G}(cn, d/n) \rightarrow P_n$ with $c = 4.86$ and $d = 7.7$. This is an improved version of a result of the authors of this paper [12] and a very similar result of Pokrovskiy [29]. Here we show that a slightly stronger bound can be obtained if random d -regular graphs are used.

Theorem 3.2. *Let $c = 5.219$ and $d = 30$. Then, a.a.s. $\mathcal{G}_{cn,d} \rightarrow P_n$, which implies that $\hat{R}(P_n) < 78.3n$ for sufficiently large n .*

Proof. Consider $\mathcal{G}_{cn,d}$ for some $c \in (2, \infty)$ and $d \in \mathbb{N}$. Our goal is to show that (for a suitable choice of c and d) the expected number of pairs of two disjoint sets, S and T , such that $|S| = |T| = n(c-2)/4$ and $e(S, T) = 0$ tends to zero as $n \rightarrow \infty$. This, together with the first moment principle, implies that a.a.s. no such pair exists and so, by Lemma 3.1, we get that a.a.s. $\mathcal{G}_{cn,d} \rightarrow P_n$. As a result, $\hat{R}(P_n) \leq (cd/2 + o(1))n$.

Let $a = a(n)$ be any function of n such that $adn \in \mathbb{Z}$ and $0 \leq a \leq (c-2)/4$, and let $X(a)$ be the expected number of pairs of two disjoint sets S, T such that $|S| = |T| = n(c-2)/4$, $e(S, T) = 0$, and $e(S, V \setminus (S \cup T)) = adn$. Using the pairing model, it is clear that

$$\begin{aligned} X(a) &= \binom{cn}{\frac{c-2}{4}n} \binom{cn - \frac{c-2}{4}n}{\frac{c-2}{4}n} \binom{\frac{c-2}{4}dn}{adn} \binom{\frac{c+2}{2}dn}{adn} M\left(\frac{c-2}{4}dn - adn\right) (adn)! \\ &\quad \cdot M\left(\frac{c+2}{2}dn - adn + \frac{c-2}{4}dn\right) / M(cdn), \end{aligned}$$

where $M(i)$ is the number of perfect matchings on i vertices, that is,

$$M(i) = \frac{i!}{(i/2)!2^{i/2}}.$$

(Each time we deal with perfect matchings, i is assumed to be an even number.) After simplification we get

$$\begin{aligned} X(a) &= (cn)! \left(\frac{c-2}{4}dn\right)! \left(\frac{c+2}{2}dn\right)! \left(\frac{3c+2}{4}dn - adn\right)! 2^{cdn/2} (cdn/2)! \\ &\quad \cdot \left[\left(\frac{c-2}{4}n\right)!^2 \left(\frac{c+2}{2}n\right)! 2^{(\frac{c-2}{4}dn - adn)/2} \left(\left(\frac{c-2}{4}dn - adn\right)/2\right)! (adn)! \right. \\ &\quad \left. \left(\frac{c+2}{2}dn - adn\right)! 2^{(\frac{3c+2}{4}dn - adn)/2} \left(\left(\frac{3c+2}{4}dn - adn\right)/2\right)! (cdn)! \right]^{-1}. \end{aligned}$$

Using Stirling's formula ($i! \sim \sqrt{2\pi i}(i/e)^i$) and focusing on the exponential part we obtain

$$X(a) = \Theta(n^{-3/2}) e^{f(a,c,d)n},$$

where

$$\begin{aligned} f(a, c, d) &= c \left(1 - \frac{d}{2}\right) \log c + \frac{c-2}{4}(d-2) \log \left(\frac{c-2}{4}\right) + \frac{c+2}{2}(d-1) \log \left(\frac{c+2}{2}\right) \\ &\quad - \left(\frac{c-2}{4} - a\right) \frac{d}{2} \log \left(\frac{c-2}{4} - a\right) - ad \log a \\ &\quad - \left(\frac{c+2}{2} - a\right) d \log \left(\frac{c+2}{2} - a\right) + \left(\frac{3c+2}{4} - a\right) \frac{d}{2} \log \left(\frac{3c+2}{4} - a\right). \end{aligned}$$

Thus, if $f(a, c, d) \leq 0$ for any integer adn under consideration, then $X(a) = O(n^{-3/2}) = o(n^{-1})$. We would get $\sum_{adn} X(a) = o(1)$ (as $adn = O(n)$), the desired property would be satisfied, and the proof would be finished.

It is straightforward to see that

$$\frac{\partial f}{\partial a} = -\frac{d}{2} \left(2 \log 2 - \log(c-2-4a) + 2 \log a - 2 \log(c+2-2a) + \log(3c+2-4a) \right).$$

Now, since $\frac{\partial f}{\partial a} = 0$ if and only if $a^2 - ca + (c^2 - 4)/8 = 0$, function $f(a, c, d)$ has a local maximum for $a = a_0 := c/2 - \sqrt{2c^2 + 8}/4$, which is also a global one on $a \in (-\infty, c/2 + \sqrt{2c^2 + 8}/4)$. Since $a \leq (c-2)/4 < c/2 + \sqrt{2c^2 + 8}/4$, we get that

$$f(a, c, d) \leq g(c, d) := f(a_0, c, d).$$

Finally, by taking $c = 5.219$ and $d = 30$, we get $g(c, d) < -0.0005$ and the proof of the first part is finished. Finally, it follows that $\hat{R}(P_n) < 78.3n$ for n large enough, as $cd/2 = 78.285 < 78.3$. (Of course, constants c and d were chosen as to minimize $cd/2$, provided that $g(c, d) \leq 0$.) \square

Lemma 3.1 provides a sufficient condition for $G \rightarrow P_n$ that is quite convenient for any good expander G . On the other hand, it is not so difficult to see that it can never give an upper bound better than $26.4n$. Indeed, let $\alpha = (c-2)/(4c)$ and G be a graph of order $N = cn$ and average degree d such that for every two disjoint sets of vertices S and T with $|S| = |T| = \alpha N$ we have $e(S, T) \neq 0$. Then the complement of G contains no copy of $K_{\alpha N, \alpha N}$ and the well-known Kővári, Sós and Turán [26] inequality (see also Theorem 11 in [8]) yields

$$N \binom{N-1-d}{\alpha N} \leq (\alpha N - 1) \binom{N}{\alpha N},$$

which for N sufficiently large implies that $d \geq \frac{\log \alpha}{\log(1-\alpha)} - 1$. Thus, the number of edges in G is at least

$$\frac{Nd}{2} = \frac{cnd}{2} \geq \frac{c}{2} \left(\frac{\log \alpha}{\log(1-\alpha)} - 1 \right) n = f(c)n,$$

where

$$f(c) := \frac{c}{2} \left(\frac{\log(c-2)/(4c)}{\log(3c+2)(4c)-1} \right).$$

The above function takes a minimum at $c = c_0 \approx 5.633$ which gives $f(c_0) \approx 26.415$.

3.2. Improved approach. In this subsection, we provide another sufficient condition for $G \rightarrow P_n$ which can be viewed as a slight straightening of Lemma 3.1. We start with the following elementary observation that is similar to the one in [12] and [29].

Lemma 3.3. *Let G be a graph of order cn for some $c > 1$. Then, the vertex set $V(G)$ can be partitioned into three sets P, U, W , $|U| = |W| = (cn - |P|)/2$ such that the graph induced by P has a Hamiltonian path and $e(U, W) = 0$.*

Proof. We perform the following algorithm on G and construct a path P . Let v_1 be an arbitrary vertex of G , let $P = (v_1)$, $U = V(G) \setminus \{v_1\}$, and $W = \emptyset$. If there is an edge from v_1 to U (say from v_1 to v_2), we extend the path as $P = (v_1, v_2)$ and remove v_2 from U . We continue extending the path P this way for as long as possible. It might happen that we reach the point of the process in which P cannot be extended, that is, there is a path from v_1 to v_k (for some $k \leq cn$) and there is no edge from v_k to U . If this is the case, v_k is moved to W and we try to continue extending the path from v_{k-1} , perhaps reaching another critical point in which another vertex will be moved to W , etc. If P is reduced

to a single vertex v_1 and no edge to U is found, we move v_1 to W and simply re-start the process from another vertex from U , again arbitrarily chosen.

An obvious but important observation is that during this algorithm there is never an edge between U and W . Moreover, in each step of the process, the size of U decreases by 1 or the size of W increases by 1. Hence, at some point of the process both U and W must have equal size, namely, $|U| = |W| = (cn - |P|)/2$. We stop the process and P, U, W form the desired partition of $V(G)$. \square

Now we are ready to state the main tool used in this subsection.

Lemma 3.4. *Let G be a graph of order cn for some $c > 2$. Assume that for every four disjoint sets of vertices S_1, S_2, T_1, T_2 such that $|S_1| + |S_2| = |T_1| + |T_2| = |S_1| + |T_1| = |S_2| + |T_2| = n(c-2)/2$ we have $e(S_1, T_2) \neq 0$ or $e(S_2, T_1) \neq 0$. Then, $G \rightarrow P_n$. (Clearly, this implies that $|S_1| = |T_2|$ and $|S_2| = |T_1|$.)*

Proof. Suppose that $G \not\rightarrow P_n$; that is, suppose that it is possible to colour the edges of G with the colours blue and red such that there is no monochromatic P_n . Let G_b be the graph on the vertex set $V(G)$, induced by blue edges. It follows from Lemma 3.3 (applied to G_b) that there exist two disjoint sets $U, W \subseteq V(G_b) = V(G)$ each of size $n(c-1)/2$ such that there is no blue edge between U and W (observe that $|P| < n$ as there is no blue P_n in G). Now, consider a bipartite graph $G_r = (U \cup W, E_r)$, with partite sets U, W , and $E_r = \{uw \in E(G) : u \in U, w \in W\}$. Clearly, all edges of G_r are red. Lemma 3.3 (this time applied to G_r) implies then that there exist two disjoint sets $U', W' \subseteq V(G_r) \subseteq V(G)$ each of size $n(c-2)/2$ such that there is no red edge between U' and W' (again, observe that $|P'| < n$ as there is no red P_n in $G \supseteq G_r$). Moreover, as G_r is bipartite, the path P' has at most $n/2$ vertices in U and at most $n/2$ vertices in W . Hence, we may assume that $|(U' \cup W') \cap U| = |(U' \cup W') \cap W| = n(c-2)/2$. Let $S_1 = U \cap U'$, $S_2 = U \cap W'$, $T_1 = W \cap U'$, and $T_2 = W \cap W'$. Clearly, $|S_1| + |S_2| = |T_1| + |T_2| = |S_1| + |T_1| = |S_2| + |T_2| = n(c-2)/2$, $e(S_1, T_2) = 0$, and $e(S_2, T_1) = 0$. The proof of the theorem is finished. \square

First, we will check how the new lemma performs for binomial random graphs.

Theorem 3.5. *Let $c = 5.28$ and $d = 6$. Then, a.a.s. $\mathcal{G}(cn, d/n) \rightarrow P_n$, which implies that $\hat{R}(P_n) < 83.7n$ for sufficiently large n .*

Proof. Consider $\mathcal{G}(cn, d/n)$. Let X be the number of (ordered) quadruples of disjoint sets S_1, S_2, T_1, T_2 such that $|S_1| + |S_2| = |T_1| + |T_2| = |S_1| + |T_1| = |S_2| + |T_2| = n(c-2)/2$ and $e(S_1, T_2) = e(S_2, T_1) = 0$. Then,

$$\mathbb{E}(X) = \binom{cn}{\frac{c-2}{2}n} \binom{cn - \frac{c-2}{2}n}{\frac{c-2}{2}n} \sum_{s=0}^{\frac{c-2}{2}n} \binom{\frac{c-2}{2}n}{s} \binom{\frac{c-2}{2}n}{\frac{c-2}{2}n - s} \left(1 - \frac{d}{n}\right)^{s^2 + (\frac{c-2}{2}n - s)^2}.$$

Since $s^2 + (\frac{c-2}{2}n - s)^2 \geq 2(\frac{c-2}{4}n)^2$ and $\sum_{s=0}^m \binom{m}{s} \binom{m}{m-s} = \binom{2m}{m}$, we get

$$\begin{aligned} \mathbb{E}(X) &\leq \binom{cn}{\frac{c-2}{2}n} \binom{cn - \frac{c-2}{2}n}{\frac{c-2}{2}n} \binom{(c-2)n}{\frac{c-2}{2}n} \left(1 - \frac{d}{n}\right)^{2(\frac{c-2}{4}n)^2} \\ &= \frac{(cn)! \cdot ((c-2)n)!}{((\frac{c-2}{2}n)!)^4 \cdot (2n)!} \left(1 - \frac{d}{n}\right)^{2(\frac{c-2}{4}n)^2} \\ &\leq \left(\frac{c^c (c-2)^{c-2}}{4 (\frac{c-2}{2})^{2(c-2)}}\right)^n e^{-\frac{d(c-2)^2}{8}n} = e^{f(c,d)n}, \end{aligned}$$

where

$$f(c, d) := c \log c + (c-2) \log(c-2) - 2 \log 2 - 2(c-2) \log((c-2)/2) - d(c-2)^2/8.$$

Observe that for $c = 5.28$ and $d = 6$, $f(c, d) < 0$ and so the first part follows by the first moment principle. Finally, it follows immediately from Chernoff's bound that the number of edges is well concentrated around $c^2 dn/2$. As $c^2 d/2 < 83.7$, we get that $\hat{R}(P_n) < 83.7n$ for n large enough. \square

As expected, random d -regular graphs give slightly better constant.

Theorem 3.6. *Let $c = 5.4806$ and $d = 27$. Then, a.a.s. $\mathcal{G}_{cn,d} \rightarrow P_n$, which implies that $\hat{R}(P_n) < 74n$ for sufficiently large n .*

Proof. Since the proof technique is exactly the same as the proof of Theorem 3.2, we only provide a sketch of the proof here. Consider $\mathcal{G}_{cn,d}$ for some $c \in (2, \infty)$ and $d \in \mathbb{N}$. Let $s = s(n)$, $a = a(n)$, $b = b(n)$, $t = t(n)$ be any integer-valued functions of n such that $0 \leq s \leq (c-2)/4$, $0 \leq a \leq s$, $0 \leq b \leq s$, $0 \leq t \leq \min\{(c-2)/2 - a - b, 2\}$. Let $X(s, a, b, t)$ be the expected number of (ordered) quadruples of disjoint sets S_1, S_2, T_1, T_2 such that $|S_1| = |T_2| = sn$, $|S_2| = |T_1| = ((c-2)/2 - s)n$, $e(S_1, T_2) = e(S_2, T_1) = 0$, $e(S_1, T_1) = adn$, $e(S_2, T_2) = bdn$, and $e(S_1 \cup S_2, V \setminus (S_1 \cup S_2 \cup T_1 \cup T_2)) = tdn$. (Note that, in particular, $|S_1| + |S_2| = |T_1| + |T_2| = |S_1| + |T_1| = |S_2| + |T_2| = n(c-2)/2$.)

Using the paring model, we get that

$$\begin{aligned} X(s, a, b, t) &= \binom{cn}{sn} \binom{(c-s)n}{(\frac{c-2}{2}-s)n} \binom{\frac{c+2}{2}n}{sn} \binom{(\frac{c+2}{2}-s)n}{(\frac{c-2}{2}-s)n} \binom{sdn}{adn} \binom{(\frac{c-2}{2}-s)dn}{adn} (adn)! \\ &\quad \cdot \binom{sdn}{bdn} \binom{(\frac{c-2}{2}-s)dn}{bdn} (bdn)! \binom{(\frac{c-2}{2}-a-b)dn}{tdn} \binom{2dn}{tdn} (tdn)! \\ &\quad \cdot M\left(\left(\frac{c-2}{2} - a - b - t\right)dn\right) M\left(\left(\frac{c+2}{2} - a - b - t\right)dn\right) / M(cdn). \end{aligned}$$

Our goal is to show that $X(s, a, b, t) = o(n^{-4})$ (regardless of the choice of s, a, b, t) so that $\sum_{s,a,b,t} X(s, a, b, t) = o(1)$. Hence, we need to maximize $X(s, a, b, t)$. One can show that the maximum is obtained for $a = b$ and for the case when $|S_1| = |S_2| = |T_1| = |T_2| = s = (c-2)/4$. Therefore, we need to concentrate on

$$Y(a, t) = X\left(\frac{c-2}{4}, a, a, t\right) = e^{f(a,t)n+o(n)},$$

where

$$\begin{aligned}
f(a, t) = & c \log c + 4(d-1) \left(\frac{c}{4} - \frac{1}{2} \right) \log \left(\frac{c}{4} - \frac{1}{2} \right) + (d-1)2 \log 2 - 2da \log a - dt \log t \\
& - \frac{d}{2} c \log c - 4d \left(\frac{c}{4} - \frac{1}{2} - a \right) \log \left(\frac{c}{4} - \frac{1}{2} - a \right) - d(2-t) \log(2-t) \\
& + d \left(\frac{c}{2} - 1 - 2a \right) \log \left(\frac{c}{2} - 1 - 2a \right) - \frac{d}{2} \left(\frac{c}{2} - 1 - 2a - t \right) \log \left(\frac{c}{2} - 1 - 2a - t \right) \\
& + \frac{d}{2} \left(\frac{c}{2} + 1 - 2a - t \right) \log \left(\frac{c}{2} + 1 - 2a - t \right).
\end{aligned}$$

Since $\frac{\partial f}{\partial t} = 0$ if and only if $t^2 - (c-4a)t + (c-2-4a) = 0$, function $f(a, t)$ has a local maximum for $t = t_0 := (c-4a)/2 - \sqrt{(c-4a)^2 - 4(c-2-4a)}/2$, which is also a global one on the interval under consideration. We get

$$f(a, t) \leq g(a) := f(a, t_0).$$

Finally, by taking $c = 5.4806$ and $d = 27$, we get $g(a) < -0.0001$ for any a we deal with. It follows that for any choice of parameters, $X(s, a, b, t) \leq Y(a, t) \leq \exp(-0.0001n) = o(n^{-4})$, and the proof is finished. It follows that $\hat{R}(P_n) < 74n$ for n large enough, as $cd/2 = 73.9881 < 74$. \square

3.3. More colours. In this subsection, we turn our attention to more than two colours. Here is a natural generalization of Lemma 3.4 in easier, bipartite, setting.

Lemma 3.7. *Let $r \in \mathbb{N} \setminus \{1\}$ and $G = (V_1 \cup V_2, E)$ be a balanced bipartite graph of order cn for some $c > 2^r - 1$. Assume that for every two sets $S \subseteq V_1$ and $T \subseteq V_2$, $|S| = |T| = ((c+1)/2^r - 1)n/2$, we have $e(S, T) \neq 0$. Then, $G \rightarrow (P_n)_r$.*

Proof. Suppose that $G \not\rightarrow (P_n)_r$; that is, suppose that it is possible to colour the edges of G with the colours from the set $\{1, 2, \dots, r\}$ such that there is no monochromatic P_n . Let β_i be defined recursively as follows: $\beta_0 = c$, $\beta_i = (\beta_{i-1} - 1)/2$ for $i \geq 1$. Note that $\beta_i = (c+1)/2^i - 1$ for $i \geq 0$. We will use (inductively) Lemma 3.3 to show the following claim, which will finish the proof (by taking $S = S_r$ and $T = T_r$).

Claim: For each $i \in \{0, 1, \dots, r\}$, there exist two sets $S_i \subseteq V_1$ and $T_i \subseteq V_2$, each of size at least $\beta_i n/2$, such that there is no edge between S_i and T_i in colour from the set $\{1, 2, \dots, i\}$.

The base case ($i = 0$) trivially (and vacuously) holds by taking $S_0 = V_1$ and $T_0 = V_2$. Suppose that the claim holds for some i , $0 \leq i < r$. We apply Lemma 3.3 to the bipartite graph with partite sets S_i, T_i , induced by the edges in colour $(i+1)$. It follows that $S_i \cup T_i$ can be partitioned into three sets P, U, W , P has a Hamiltonian path, $|U| = |W| = (\beta_i n - |P|)/2$, and $e(U, W) = 0$. Since G is bipartite, $|S_i \setminus P| = |T_i \setminus P| = (\beta_i n - |P|)/2$. Without loss of generality, we may assume that $|(S_i \setminus P) \cap U| \geq |(T_i \setminus P) \cap U|$. As a result, $|(S_i \setminus P) \cap U| = |(T_i \setminus P) \cap W| \geq n(\beta_i - |P|)/4 \geq n(\beta_i - 1)/4$. The inductive step is finished by taking $S_{i+1} = (S_i \setminus P) \cap U$ and $T_{i+1} = (T_i \setminus P) \cap W$. \square

Theorem 3.8. *Let $r \in \mathbb{N} \setminus \{1\}$, $c = 2^{r+1}$, and $d = 8r$. Then, a.a.s. $\mathcal{G}(cn, cn, d/n) \rightarrow (P_n)_r$, which implies that $\hat{R}(P_n, r) < 33r4^r n$ for sufficiently large n .*

Proof. Consider $\mathcal{G}(cn, cn, d/n) = (V_1 \cup V_2, E)$. We will show that the expected number of pairs of sets $S \subseteq V_1$ and $T \subseteq V_2$ such that $|S| = |T| = cn/2^{r+2}$ and $e(S, T) = 0$ tends to zero as $n \rightarrow \infty$. This will finish the first part of the proof by Lemma 3.7, combined with the first moment principle, as $cn/2^{r+2} < ((c+1)/2^r - 1)n/2$ (recall that $c = 2^{r+1}$). Indeed, the expectation we need to estimate is equal to

$$\begin{aligned} \binom{cn}{cn/2^{r+2}}^2 \left(1 - \frac{d}{n}\right)^{(cn/2^{r+2})^2} &\leq (2^{r+2}e)^{2cn/2^{r+2}} \exp\left(-d \left(\frac{c}{2^{r+2}}\right)^2 n\right) \\ &= o\left((e^{2r})^{2cn/2^{r+2}} \exp\left(-d \left(\frac{c}{2^{r+2}}\right)^2 n\right)\right) \\ &= o\left(\exp\left(\left(4r - \frac{dc}{2^{r+2}}\right) \frac{cn}{2^{r+2}}\right)\right) = o(1), \end{aligned}$$

as $dc/2^{r+2} = 4r$. The second part follows from the fact that the number of edges in $\mathcal{G}(cn, cn, d/n)$ is well concentrated around $c^2 dn$ and $c^2 d = 32r4^r < 33r4^r$. \square

Summarizing, we showed that there exist some positive constants c_1, c_2 such that for any $r \in \mathbb{N}$ we have

$$c_1 r^2 \cdot n \leq \hat{R}(P_n, r) \leq c_2 r 4^r \cdot n.$$

Of course, one can improve Lemma 3.7 slightly. For example, in the first step there is no need to assume that the graph is bipartite. Also one could try to use the “double wholes” approach as in Lemma 3.4. However, the improvement would not be substantial. It would be interesting to determine the order of magnitude of $\hat{R}(P_n, r)$ as a function of r (for fixed n).

4. MULTICOLOURED PATH RAMSEY NUMBER OF $\mathcal{G}(n, p)$

Determining the classical Ramsey number for paths, $R(P_n, r)$, it is a well-known problem that attracted a lot of attention. The case $r = 2$ is well understood, due to the result of Gerencsér and Gyárfás [19]. It is known that

$$R(P_n, 2) = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

For $r = 3$ and n sufficiently large, Gyárfás, Ruszinkó, Sárközy, and Szemerédi [21, 22] proved that

$$R(P_n, 3) = \begin{cases} 2n-1 & \text{for odd } n, \\ 2n-2 & \text{for even } n, \end{cases}$$

as conjectured earlier by Faudree and Schelp [15]. (An asymptotic value was obtained earlier by Figaj and Łuczak [16].) However, this problem is still open for *small* values of n . On the other hand, very little is known for any integer $r \geq 4$. The well-known Erdős and Gallai result [14] (see Theorem 4.2 below) implies only that $R(P_n, r) \leq rn$. Very recently, Sárközy [30] improved it and showed that for any integer $r \geq 2$,

$$R(P_n, r) \leq \left(r - \frac{r}{16r^3 + 1}\right)n.$$

It is believed that the value of $R(P_n, r)$ is close to $(r-1)n$.

In this section, we consider an analogous problem for $\mathcal{G}(n, p)$ with average degree, np , tending to infinity as $n \rightarrow \infty$. We are interested in the following constant:

$$c_r = \sup\{c \in [0, 1] : \mathcal{G}(n, p) \rightarrow (P_{cn})_r \text{ a.a.s., provided } np \rightarrow \infty\}. \quad (1)$$

The case $r = 2$ is already investigated; due to Letzter [27] we know that $c_2 = 2/3$. For any integer $r \geq 3$, Lemma 3.7 gives only $c_r \geq 1/(2^r - 1)$. We will show a stronger result.

Theorem 4.1. *Let $r \in \mathbb{N} \setminus \{1, 2, 3\}$, $\alpha > 0$ be an arbitrarily small constant, and $p = p(n)$ be such that $pn \rightarrow \infty$. Then, a.a.s. $\mathcal{G}(n, p) \rightarrow (P_{(1/r-\alpha)n})_r$, which implies that $c_r \geq 1/r$. Furthermore, for 3 colours, a.a.s. $\mathcal{G}(n, p) \rightarrow (P_{(1/2-\alpha)n})_3$, which is optimal and implies that $c_3 = 1/2$.*

Furthermore, we conjecture that $c_r = n/R(P_n, r)$ for any $r \geq 2$, which is true for $r = 2$ [27] and for $r = 3$, due to the above theorem.

First we prove Theorem 4.1 for $r \geq 4$. Let us start with the Erdős and Gallai result [14] and its perturbed version.

Theorem 4.2 ([14]). *Let G be a graph of order n with no P_k . Then, $|E(G)| \leq n(k-2)/2$.*

After applying this theorem to the subgraph of G induced by the majority colour, we get the following corollary.

Corollary 4.3. *Let $r \in \mathbb{N} \setminus \{1, 2\}$ and $0 < \varepsilon < 1$. Then, for every graph G of order n with at least $(1-\varepsilon)\binom{n}{2}$ edges we have $G \rightarrow (P_k)_r$, where $k = (1-\varepsilon)n/r$.*

Now we introduce some notation needed to state Sparse Regularity Lemma. For given two disjoint subsets of vertices U and W in a graph G , we define the p -density of the edges between U and W as

$$d_p(U, W) = \frac{e(U, W)}{p|U||W|}.$$

Moreover, we say that U, W is an (ε, p) -regular pair if, for every $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon|U|$, $|W'| \geq \varepsilon|W|$, $|d_p(U', W') - d_p(U, W)| \leq \varepsilon$. Suppose that $0 < \eta < 1$, $D > 1$ and $0 < p < 1$ are given. We will say that a graph G is (η, p, D) -upper-uniform if for all disjoint subsets U_1 and U_2 with $|U_1| \geq |U_2| \geq \eta|V(G)|$, $d_p(U_1, U_2) \leq D$.

The following theorem, which is a variant of Szemerédi's Regularity Lemma [32] for sparse graphs, was discovered independently by Kohayakawa [25] and Rödl (see, for example, [10]).

Theorem 4.4 (Sparse Regularity Lemma). *For every $\varepsilon > 0$, $r \geq 1$ and $D \geq 1$, there exist $\eta > 0$ and T such that for every $0 \leq p \leq 1$, if G_1, G_2, \dots, G_r are (η, p, D) -upper-uniform graphs on the vertex set V , then there is an equipartition of V into s parts, where $1/\varepsilon \leq s \leq T$, for which all but at most $\varepsilon\binom{s}{2}$ of the pairs induce an (ε, p) -regular pair in each G_i .*

Now, we are ready to prove the main theorem of this section. Recall that for $r = 2$, Letzter [27] showed that $c_2 = 2/3$. The proof below is essentially her approach that easily extends to any numbers of colours.

Proof of Theorem 4.1 for $r \geq 4$. Let $r \in \mathbb{N} \setminus \{1, 2, 3\}$, $\alpha > 0$, and $p = p(n)$ be such that $pn \rightarrow \infty$ as $n \rightarrow \infty$. We will show that a.a.s. for every r -edge colouring of $G = \mathcal{G}(n, p) = (V, E)$ there is a monochromatic path of length at least $(1/r - \alpha)n$.

Pick $\varepsilon = \varepsilon(\alpha) > 0$ such that $(1 - 8\varepsilon)(1 - (r + 1)\varepsilon) \geq 1 - r\alpha$ and $1/(2r) > \varepsilon$ and set $D = 2$. Apply the sparse regularity lemma with above defined ε, D , and r . Let η and T be the constants arising from this lemma.

For each $i \in [r]$, let G_i be a subgraph of G induced by the edges coloured with colour i . By Chernoff's bound, for any U and W of size at least ηn , the p -density $d_p(U, W)$ in G is at most 2 and so the p -density in each G_i is also at most 2. (Indeed, there are obviously at most $(2^n)^2 = 4^n$ choices for U and W , and for each choice the failure probability is at most $2 \exp(-\eta^2 n^2 p/3) = o(4^n)$.) Thus, each G_i is an (η, p, D) -upper-uniform graph. Consequently, Theorem 4.4 implies that there is an equipartition of $V = V_1 \cup V_2 \cup \dots \cup V_s$, where $1/\varepsilon \leq s \leq T$, for which all but at most $\varepsilon \binom{s}{2}$ of the pairs induce an (ε, p) -regular pair in each G_i .

Let R be the auxiliary (cluster) graph with vertex set $[s]$, where $\{i, j\}$ is an edge if and only if V_i, V_j induce an (ε, p) -regular bipartite graph in each of the r colours. Colour $\{i, j\}$ in R by the majority colour appearing between V_i and V_j in G . Again by Chernoff's bound the p -density $d_p(V_i, V_j)$ in G is at least $1/2$. Hence, if $\{i, j\}$ is coloured by c , then $d_p(V_i, V_j)$ in G_c is at least $1/(2r)$.

Observe that the number of edges in R is at least $(1 - \varepsilon) \binom{s}{2}$. Hence, it follows from Corollary 4.3 that R contains a monochromatic, say red, path $P = (i_1, i_2, \dots, i_\ell)$ on at least $\ell = (1 - \varepsilon)s/r$ vertices. Furthermore, we divide each set V_{i_j} into two sets U_j, W_j of equal sizes, that is, $|U_j| = |W_j| = n/(2s)$. Let P_j be a longest red path in the bipartite graph $G[U_j, W_{j+1}]$. Since V_{i_j} and $V_{i_{j+1}}$ are (ε, p) -regular with p -density at least $1/(2r)$, Lemma 3.3 implies that P_j covers at least $(1 - 4\varepsilon)n/s$ vertices of $G[U_j, W_{j+1}]$ for each $1 \leq j \leq \ell - 1$.

Now, we are going to glue $P_1, P_2, \dots, P_{\ell-1}$, trying to lose as few vertices as possible. Let X_j be the last $\varepsilon n/s$ vertices of P_j in U_j , and let Y_{j+1} be the first $\varepsilon n/s$ vertices of P_{j+1} in U_{j+1} . Since V_j, V_{j+1} is an (ε, p) -regular pair (in the graph induced by red edges) with p -density at least $1/(2r) > \varepsilon$, there must be a red edge between X_j and Y_{j+1} . Thus, G has a red path Q which contains all vertices of $V(P_1) \cup V(P_2) \cup \dots \cup V(P_{\ell-1})$ but at most $4\varepsilon(\ell - 1)n/s$. Consequently,

$$\begin{aligned} |V(Q)| &\geq (\ell - 1)(1 - 4\varepsilon)n/s - 4\varepsilon(\ell - 1)n/s = (1 - 8\varepsilon)(\ell - 1)n/s \\ &\geq (1 - 8\varepsilon)(1 - (r + 1)\varepsilon)n/r \geq (1/r - \alpha)n, \end{aligned}$$

as required. \square

Now we show how to prove Theorem 4.1 for $r = 3$. The proof is based on an ingenious idea of Figaj and Łuczak [16] of “connected matchings” and relies on the following lemma.

Theorem 4.5 ([16]). *Let $0 < \varepsilon \leq 0.001$ and let G be a graph of order n with at least $(1 - \varepsilon) \binom{n}{2}$ edges. Then, for any 3-colouring of the edges of G , there is a monochromatic component which contains a matching saturating at least $(1/2 - 5\varepsilon^{1/7})n$ vertices.*

Proof of Theorem 4.1 for $r = 3$. The proof is very similar to the case $r \geq 4$. Therefore, we only emphasize differences. As in the previous case we apply the sparse regularity lemma to an r -coloured graph G and then Theorem 4.5 to the r -coloured cluster graph R of order

s . This way we obtain a monochromatic, say red, minimal component F which contains a matching M saturating at least $\ell = (1/2 - 5\varepsilon^{1/7})s$ vertices of R . Let $W = (i_1, i_2, \dots, i_k)$ be a minimal walk contained in F which contains M . Clearly, F is a tree and so $k \leq 2(s-1)$. For each $e \in M$ we find the first appearance of e in W , say (i_j, i_{j+1}) , and replace it by a red path P_j of length $(1 - 4\varepsilon)n/s$ which alternates between V_{i_j} and $V_{i_{j+1}}$ in G . Clearly,

$$\sum_{j=1}^{\ell} |P_j| \geq \ell \cdot (1 - 4\varepsilon)n/s = (1/2 - 5\varepsilon^{1/7})(1 - 4\varepsilon)n.$$

Finally using elementary properties of (ε, p) -regular pairs we glue all P_j 's (following the order in W) as in the previous case losing only $\text{poly}(\varepsilon)n$ vertices. \square

It immediately follows from the above proof that a better constant in Corollary 4.3 yields a bigger value of c_r .

5. LARGE MONOCHROMATIC COMPONENTS IN $\mathcal{G}(n, p)$

It is easy to see that in every 2-colouring of the edges of K_n there is a monochromatic connected subgraph on n vertices. For three colours the analogue problem was first solved by Gerencsér and Gyárfás [19] (see also [1, 4]). The generalization of this result to any number of colours was proved by Gyárfás [20] and it also follows from a more general result of Füredi [18].

Theorem 5.1 ([20, 18]). *Let $r \in \mathbb{N} \setminus \{1\}$. Suppose that the edges of K_n are coloured with r colours. Then, there is a monochromatic component with at least $n/(r-1)$ vertices. This result is sharp if $r-1$ is a prime power and $(r-1)^2$ divides n .*

In this section we consider a similar problem for $\mathcal{G}(n, p)$. The following was proven by Spöhel, Steger and Thomas [31] and also independently by Bohman, Frieze, Krivelevich, Loh and Sudakov [5].

Theorem 5.2 ([31, 5]). *Let $r \in \mathbb{N} \setminus \{1\}$ and let τ_r denote the constant which determines the threshold for r -orientability of the random graph $\mathcal{G}(n, rc/n)$. Then, for any constant $c > 0$ the following holds a.a.s.*

- (i) *If $c < \tau_r$, then there exists an r -colouring of the edges of $\mathcal{G}(n, rc/n)$ in which all monochromatic components have $o(n)$ vertices.*
- (ii) *If $c > \tau_r$, then every r -colouring of the edges of $\mathcal{G}(n, rc/n)$ contains a monochromatic component with $\Theta(n)$ vertices.*

Here we complement this result considering the case when the average degree tends to infinity (as $n \rightarrow \infty$). This time, we are interested in the following constant:

$$d_r = \sup\{d \in [0, 1] : \mathcal{G}(n, p) \text{ has a monochromatic component on at least } dn \text{ vertices a.a.s., provided } np \rightarrow \infty\}.$$

Clearly $d_r \geq c_r$, where c_r is defined as in the previous section (cf. (1)).

Theorem 5.3. *Let $r \in \mathbb{N} \setminus \{1\}$, $\alpha > 0$ be an arbitrarily small constant, and $p = p(n)$ be such that $pn \rightarrow \infty$. Then, a.a.s. for any r -colouring of the edges of $\mathcal{G}(n, p)$ there is a monochromatic component on at least $(1/(r-1) - \alpha)n$ vertices, which implies that $d_r = 1/(r-1)$. This constant is optimal for infinitely many r .*

First we derive an analogous result to Corollary 4.3, which is a perturbed version of Theorem 5.1.

Lemma 5.4. *Let $r \in \mathbb{N} \setminus \{1\}$ and $0 < \varepsilon \leq 1/r^2$. Let G be a graph of order n with at least $(1 - \varepsilon)\binom{n}{2}$ edges. Then, for any r -colouring of the edges of G there is a monochromatic component on at least $(1/(r - 1) - \varepsilon r^2)n$ vertices.*

Let us note that a special case of this result for $r = 3$ was obtained by Figaj and Łuczak [16]. Our proof is different; we will use the following result of Liu, Morris and Prince [28].

Lemma 5.5 (Lemma 9 in [28]). *Let $H = (V_1, V_2, E)$ be a bipartite graph. Assume that $|E| \geq \eta|V_1||V_2|$ for some $\eta > 0$. Then, H has a component on at least $\eta(|V_1| + |V_2|)$ vertices.*

Proof of Lemma 5.4. Let $G = (V, E)$ be a graph of order n with at least $(1 - \varepsilon)\binom{n}{2} \geq \binom{n}{2} - (\varepsilon/2)n^2$ edges. For a contradiction, suppose that there is a colouring of the edges of G with r colours so that C , a largest monochromatic component in G , satisfies $|V(C)| < (1/(r - 1) - \varepsilon r^2)n$. On the other hand, by Corollary 4.3, $|V(C)| \geq (1/r - \varepsilon)n$.

Consider the bipartite graph F induced by the edges of G between $V(C)$ and $V(G) \setminus V(C)$. Clearly, the edges of F are coloured with at most $r - 1$ colours (as the colour of C is not used). First observe that

$$\begin{aligned} (\varepsilon/2)n^2 &= |V(C)||V(G) \setminus V(C)| \cdot \frac{\varepsilon n^2}{2|V(C)||V(G) \setminus V(C)|} \\ &\leq |V(C)||V(G) \setminus V(C)| \cdot \frac{\varepsilon n^2}{2(1/r - \varepsilon)n \cdot (1 - (1/r - \varepsilon))n}. \end{aligned}$$

Since $\varepsilon \leq 1/r^2$ and $r \geq 2$, we get $1/r - \varepsilon = (1 - \varepsilon r)/r \geq (1 - 1/r)/r \geq 1/(2r)$. Thus,

$$(\varepsilon/2)n^2 \leq |V(C)||V(G) \setminus V(C)| \cdot \frac{\varepsilon}{2 \cdot 1/(2r) \cdot (r - 1)/r} \leq |V(C)||V(G) \setminus V(C)| \cdot \varepsilon r^2.$$

Consequently,

$$|E(F)| \geq |V(C)||V(G) \setminus V(C)| - (\varepsilon/2)n^2 \geq (1 - \varepsilon r^2)|V(C)||V(G) \setminus V(C)|.$$

Let H be a subgraph of F induced by the majority colour. Thus,

$$|E(H)| \geq \frac{1}{r - 1}(1 - \varepsilon r^2)|V(C)||V(G) \setminus V(C)|,$$

and so Lemma 5.5 implies that there is a monochromatic component of order

$$\frac{1}{r - 1}(1 - \varepsilon r^2)n \geq \left(\frac{1}{r - 1} - \varepsilon r^2 \right) n,$$

that is larger than C , a largest monochromatic component in G . We get the desired contradiction and the proof is finished. \square

Finally, we are ready to sketch the proof of the main result of this section.

Sketch of the proof of Theorem 5.3. This is basically the proof of Theorem 4.1 with Corollary 4.3 replaced by Lemma 5.4. We find a monochromatic spanning tree on $(1/(r - 1) - \varepsilon r^2)s$ vertices in the cluster graph, and then we replace each edge by a long path (in a bipartite graph). All those paths intersect yielding a large monochromatic component. The sharpness follows immediately from the sharpness of Theorem 5.1. \square

6. CONCLUDING REMARKS

We finish the paper with a few remarks and possible questions for future work. In this paper, we improved both a lower and an upper bound for $\hat{R}(P_n)$, but clearly there is still a lot of work that is waiting to be done. Closing the gap is a natural question. However, it seems that in order to obtain a substantial improvement, one needs to develop a new approach to attack this question. For more colours, as we already mentioned, it is interesting to determine the order of magnitude of $\hat{R}(P_n, r)$ as a function of r . Is it exponential in r ? Or maybe it is only polynomial in r ?

In this paper, we are also concerned with monochromatic paths and components in $\mathcal{G}(n, p)$, provided that $pn \rightarrow \infty$. Exactly the same question can be asked for $\mathcal{G}_{n,d}$. It is known, due to a result of Kim and Vu [24], that if $d \gg \log n$ and $d \ll n^{1/3}/\log^2 n$, then there exists a coupling of $\mathcal{G}(n, p)$ with $p = \frac{d}{n}(1 - (\log n/d)^{1/3})$, and $\mathcal{G}_{n,d}$, such that a.a.s. $\mathcal{G}(n, p)$ is a subgraph of $\mathcal{G}_{n,d}$. A recent result of Dudek, Frieze, Ruciński, and Šileikis [11] (see also Section 10.3 in [17]) extends that for denser graphs. Consequently, our results for $\mathcal{G}(n, p)$ model imply immediately the counterpart results for $\mathcal{G}_{n,d}$, provided $d \gg \log n$. It would be interesting to investigate the behaviour for $\Omega(1) = d = O(\log n)$.

Finally, determining the value of c_r might be of some interest (cf. (1)). Letzter [27] showed that $c_2 = 2/3$ and in this paper we showed that $c_3 = 1/2$. For $r \in \mathbb{N} \setminus \{1, 2, 3\}$ we proved that $1/r \leq c_r \leq 1/(r-1)$ but the exact value of c_r still remains unknown.

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